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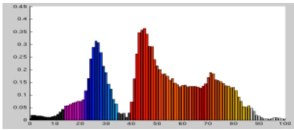


Signals and Systems
Lecture 14- Fourier Series Representation of Periodic Signals (Part 2)

Outline
- Examples

Examples

Example 1:

Find the Fourier series coefficients (exponential form) for the signal

$$x(t) = cos\,(\omega_0 t)$$

Solution:

$$D_k = \frac{1}{T}\int_{\langle T\rangle} x(t).\,e^{-jk\omega_0 t}\,dt$$

$$D_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^T cos(\omega_0 t)dt = 0$$

$$D_k = \frac{1}{T}\int_0^T cos(\omega_0 t).\,e^{-jk\omega_0 t}dt =$$

$$= \frac{1}{2T}\int_0^T (e^{j\omega_0 t} + e^{-j\omega_0 t}).\,e^{-jk\omega_0 t}dt =$$

$$= \frac{1}{2T}\left[\int_0^T e^{j\omega_0(1-k)t}dt \quad + \quad \int_0^T e^{-j\omega_0(1+k)t}dt\right]$$

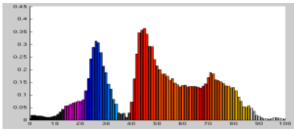
$$= \begin{cases} \frac{1}{2} & if \quad k = 1 \\ \frac{1}{2} & if \quad k = -1 \\ 0 & if \quad k \neq \pm 1 \end{cases}$$

$$x(t) = cos(\omega_0 t) = \frac{1}{2}.\,e^{j\omega_0 t} + \frac{1}{2}.\,e^{-j\omega_0 t}$$

Homework:

Find the Fourier series coefficients (exponential form) for the signal

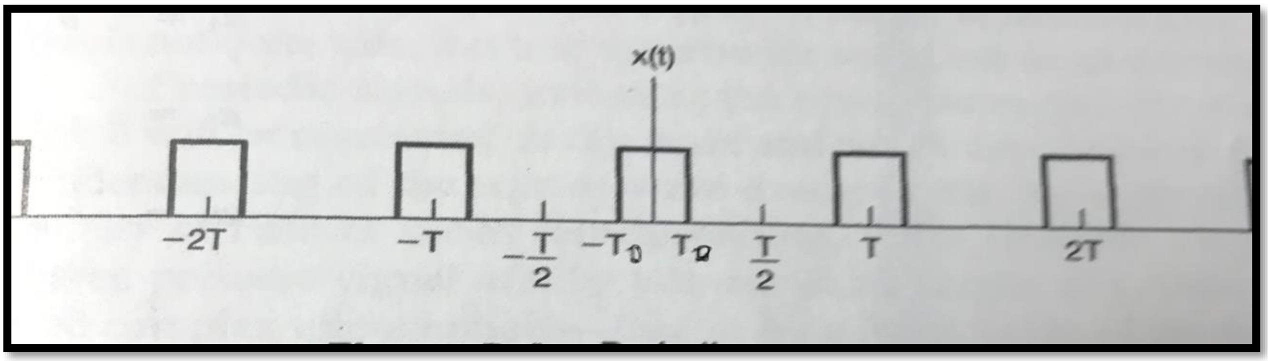
$$x(t) = sin\,(\omega_0 t)$$



Example 2:

Determine the FS representation (exponential form) of the square wave depicted in the next figure.

$$x(t) = \begin{cases} 1 & |t| < T_0 \\ 0 & T_0 < |t| < \dfrac{T}{2} \end{cases}$$



Solution:

The period is T, so $\omega_0 = \frac{2\pi}{T}$, because the $x(t)$ has symmetry, it's simpler to evaluate by integrating over the range $-T/2 \le t \le T/2$, we obtain:

$$D_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t).e^{-jk\omega_0 t}\,dt, \quad \textit{Assume that } \; T = 4T_0$$

$$= \frac{1}{T}\int_{-T_0}^{T_0} x(t).e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_0}^{T_0} 1.e^{-jk\omega_0 t}\,dt$$

$$D_0 = \frac{1}{T}\int_{-T_0}^{T_0} dt = \frac{2T_0}{T} = \frac{2T_0}{4T_0} = \frac{1}{2}$$

$$D_k = \frac{1}{T}\int_{-T_0}^{T_0} 1.e^{-jk\omega_0 t}\,dt = \frac{-1}{jTk\omega_0}.e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right] = \frac{2\,\sin(k\omega_0 T_0)}{k\omega_0 T}$$

$$D_k = \frac{\sin(k\omega_0 T_0)}{k\pi} = \frac{\sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

$$\textit{where } \omega_0 T = 2\pi$$

$$\textit{and } \; \omega_0 T_0 = \frac{\pi}{2}, \quad \Leftarrow \quad \frac{2\pi T_0}{T} = \frac{2\pi T_0}{4T_0} = \frac{\pi}{2}$$

$$D_k = \frac{\sin\left(\frac{\pi k}{2}\right)}{k\pi}$$

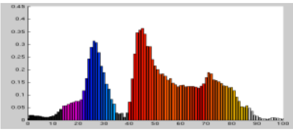
$D_1 = D_{-1} = 1/\pi$

$D_3 = D_{-3} = -1/3\pi$

$D_5 = D_{-5} = 1/5\pi$

And

$D_k = 0 \; \textit{for } k \textit{ even}$

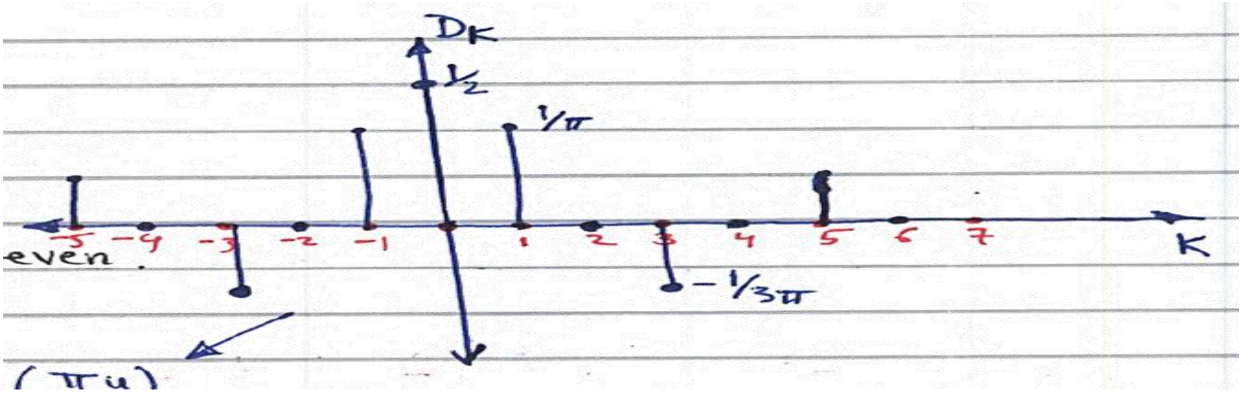


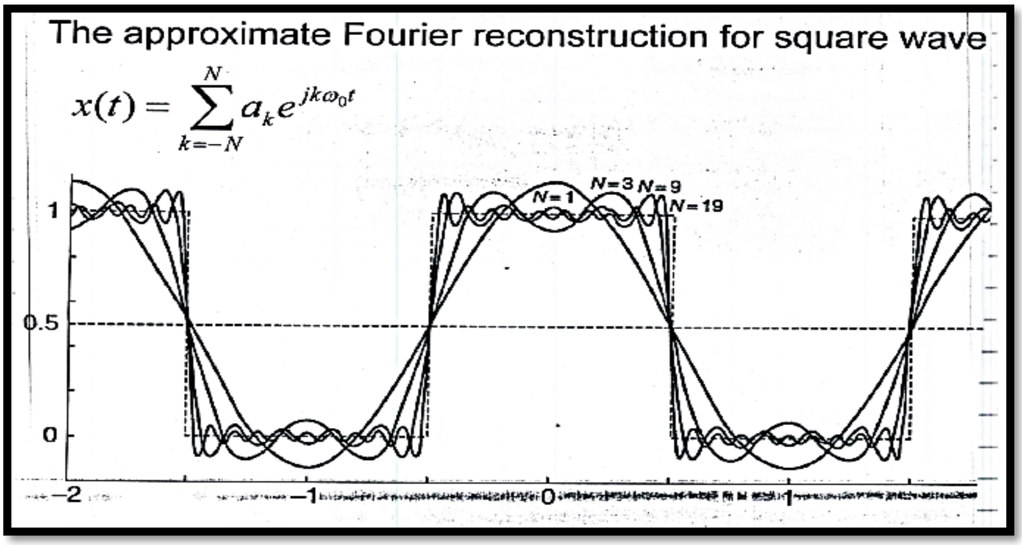
❋ Normalized sinc

$$sinc(x) = \begin{cases} 1 & x = 0 \\ \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \end{cases}$$

❋ Unnormalized sinc

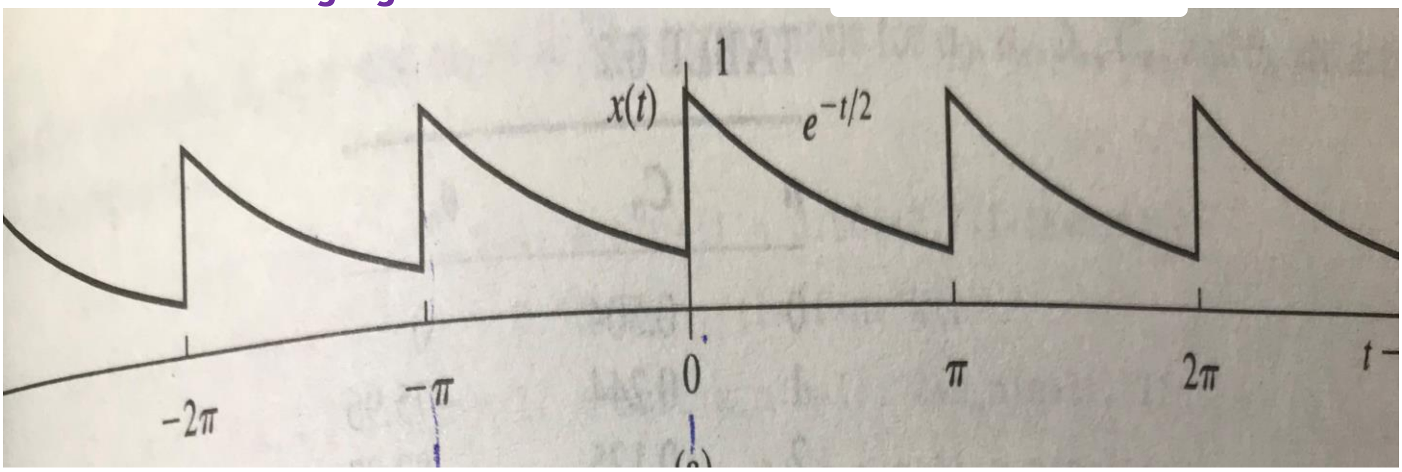
$$sinc(x) = \begin{cases} 1 & x = 0 \\ \dfrac{\sin(x)}{x} & x \neq 0 \end{cases}$$

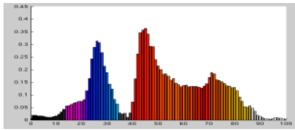




Example 3:

a) **Find the compact trigonometric FS for the Periodic signal $x(t)$ shown in the following figure.**





b) Sketch the amplitude and phase spectra of $x(t)$

Solution:

a) In this, case period $T_0 = \pi$ and the fundamental and frequency

$$f_0 = \frac{1}{T_0} = \frac{1}{\pi}\ Hz \quad and \quad \omega_0 = \frac{2\pi}{T} = 2\ rad/s$$

Therefore:

$$x(t) = a_0 + \sum_{k=1}^{\infty}[a_k \cos(k\omega_0 t) + b_k\ \sin(k\omega_0 t)]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}[a_k . \cos(2kt) + b_k\ \sin(2kt)]$$

$$where \quad a_0 = \frac{1}{\pi}.\int_{T_0} x(t)dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}dt = 0.504$$

$$a_k = \frac{2}{\pi}.\int_0^{\pi} e^{-t/2}.\cos(2kt)\,dt = 0.504\left(\frac{2}{1+16k^2}\right)$$

And

$$b_k = \frac{2}{\pi}.\int_0^{\pi} e^{-t/2}.\sin(2kt)\,dt = 0.504\left(\frac{8k}{1+16k^2}\right)$$

$$x(t) = 0.504\left[1 + \sum_{k=1}^{\infty}\left(\frac{2}{1+16k^2}\right)(\cos(2kt) + 4k.\sin(2kt))\right]$$

Trigonometric FS form (1)

❋ **Compact trigonometric FS:**

$$c_0 = a_0 = 0.504$$

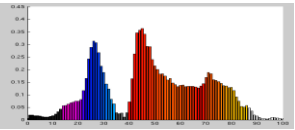
$$c_k = \sqrt{a_k^2 + b_k^2} = 0.504\sqrt{\frac{4}{(1+16k^2)^2} + \frac{64k^2}{(1+16k^2)^2}}$$

$$= 0.504\left(\frac{2}{\sqrt{1+16k^2}}\right)$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right) = \tan^{-1}(-4k) = -\tan^{-1}(4k)$$

❋ **Amplitude and phases of the dc and first seven harmonics are computed from the above equations, and displayed in the following table:**

k	C_k	θ_k
0	0.504	0
1	0.244	-75.96°
2	0.125	-82.87°
3	0.084	-85.24°
4	0.063	-86.42°
5	0.0504	-87.14°
6	0.042	-87.61°
7	0.036	-87.95°



❋ **We can use these numerical values to express $x(t)$ as:**

$$x(t) = 0.504 + 0.504 \sum_{k=1}^{\infty} \sqrt{\frac{2}{1+16k^2}} \; cos(2kt - tan^{-1}(4k))$$

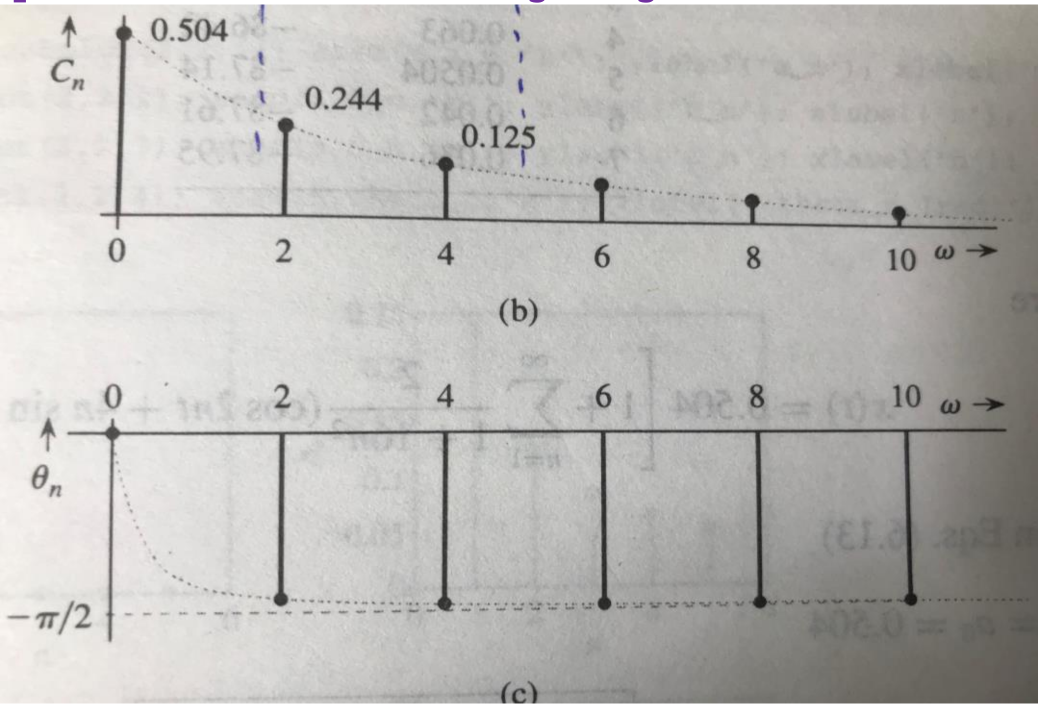
$$x(t) = 0.504 + 0.244\, cos(2t - 75.96°) + 0.125\, cos(4t - 82.87°) + 0.084\, cos(6t - 85.24°) + 0.063\, cos(8t - 86.42°) + \cdots$$

b) Plot the frequency spectra

❋ **The frequency spectra provide an alternative description- the frequency domain description of $x(t)$.**

❋ **A signal, therefore has a dual identity:**

- **The time domain identity $x(t)$. and the frequency- domain identity (Fourier Spectra).**
- **The two identities complement each other, taken together; they provide better understanding of signal.**



Fourier Spectra = Magnitude Spectrum + phase spectrum